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LETTER TO THE EDITOR

A fully consistent Lie algebraic representation of quantum phase and number operators

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Abstract
A fully consistent realization of the quantum operators corresponding to the canonically conjugate phase and number variables is proposed, resorting to the $\kappa = \frac{1}{2}$ positive discrete series of the irreducible unitary representation of the Lie algebra $su(1,1)$ of the double covering group of $SO^{\uparrow}(1,2)$. The realization holds in subspace $\mathfrak{F}\backslash|0\rangle$, the system Fock space minus the vacuum state. A possible way to extend it to the full space of states based on recourse to a dilated extension of Hilbert space is discussed.

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The original representation of quantum mechanics of Bohr and Sommerfeld implicitly relies on the notion of integrable dynamical system, the correspondence principle referring to classical canonically conjugate action-angle variables. The naturally ensuing question concerning the existence of a quantum operator related to the classical observable 'phase', that would put phase and action on the same footing as position and momentum of the more widely adopted Schrödinger and Heisenberg formulations, was posed by London [1] as early as 1926. However the practical and operational relevance of a consistent quantum mechanical description of phase emerged only much later, with the onset of quantum optics [2, 3]. Here, in particular, the notion as well as the experimental realization of squeezed states of light—allowing us to perform high accuracy measurements of the interferometric phase—made the difference, promising to open the way towards the capability of engineering arbitrary states of the radiation field [4]. The concept of phase operator has also been crucial to the intriguing operational approach proposed by Noh, Fougères and Mandel in a series of papers between 1991 and 1993 (essentially summarized in [3]) and thoroughly discussed in [5], where an interesting historical view of the problem is presented.

Nowadays the question of consistently representing field phase and intensity operators has become crucial not only for quantum optics but also for the physics of mesoscopic quantum systems, such as, for instance, superconductive Josephson junctions [6, 7], and weakly interacting Bose–Einstein condensates with Josephson coupling [8, 9], of wide interest for quantum information manipulation [10–13]. The problem of how to describe the canonically conjugate phase $\hat{\phi}$ and number $\hat{n}$ operators ($[\hat{\phi}, \hat{n}] = i$) as self-adjoint operators in a formalism consistent with the representation of amplitudes through creation and annihilation operators in the Hilbert (e.g. Fock) space $\mathfrak{F}$ of states of the corresponding physical system[1] remains however essentially unsolved. The several attempts made over the years [15–24] have clarified a number of interesting features and have proposed practical approaches to deal with various obstructions, but none has fully overcome the difficulties, inherent essentially to the inconsistency related with the definition of the phase operator (whose eigenvalues provide the phase difference with respect to an arbitrary reference value ϕ_0) in the state where the number eigenvalue is zero (a difficulty which has a classical counterpart [25]).

The correspondence principle of quantum mechanics appears to suggest that the customary annihilation and creation operators $a, a^\dagger$ (related to the complex classical amplitude of harmonic oscillations) adopted, e.g., in the description of the harmonic oscillator, which together with the number $\hat{n}$ and identity operator $\mathbb{I}$, generate the Weyl–Heisenberg algebra $h(1)$ ($[a, a^\dagger] = \mathbb{I}, [a, \hat{n}] = a, [a^\dagger, \hat{n}] = -a^\dagger, [\bullet, \mathbb{I}] = 0$), may be represented in polar form (phase-modulus decomposition) as

$$a \doteq \hat{e}\sqrt{\hat{n}}, \tag{1}$$

$\hat{e}$ defining the quantum phase operator. However, it is not possible to derive from the above ansatz a fully satisfactory phase observable: indeed, the solution of (1) $\hat{e} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|$, where $\hat{n}|n\rangle = n|n\rangle$, $n \geqslant 0$, turns out to be non-unitary;

$$\hat{e}\hat{e}^\dagger = \mathbb{I}, \qquad \hat{e}^\dagger\hat{e} = \mathbb{I} - |0\rangle\langle 0|. \tag{2}$$

The approach to the problem most adopted in the applications is that due to Pegg and Barnett [22], who propose to resort to a finite-dimensional [14] Hilbert space of states $\mathcal{H}_N \doteq \mathrm{span}\{|n\rangle | n = 0, \ldots, N-1\}$ ($N = \dim(\mathcal{H}_N)$) and to define in it a self-adjoint phase operator $\hat{\phi}_N$ by constructing first the set of 'phase states'

$$|\phi_\ell\rangle \doteq N^{-\frac{1}{2}} \sum_{n=0}^{N-1} \exp(\mathrm{i}n\phi_\ell)|n\rangle, \qquad \phi_\ell \doteq \phi_0 + 2\pi \frac{\ell}{N}, \tag{3}$$

for $\ell \in \mathbb{Z}_N$, and setting then $\hat{\phi}_N \doteq \sum_{\ell=0}^{N-1} \phi_\ell |\phi_\ell\rangle\langle\phi_\ell|$. In the occupation number basis $\hat{\phi}_N$ has matrix elements

$$\langle\ell|\hat{\phi}_N|\ell\rangle = \phi_0 + \pi(N-1)N^{-1} \qquad \text{and,} \qquad \text{for} \quad m \neq \ell,$$
$$\langle m|\hat{\phi}_N|\ell\rangle = \frac{2\pi^2}{N}\left|\sin\left(\frac{\pi}{N}(m-\ell)\right)\right|^{-1} \mathrm{e}^{\mathrm{i}(m-\ell)(\phi_0 + \frac{1}{N}\pi)}.$$

The problem here is that the spectral resolution of the discrete operator $\hat{\phi}_N$ does not provide [26] a measure converging either to a projection valued measure or to a probability operator measure in the limit $N \to \infty$ (i.e. in the full Hilbert–Fock space $\mathfrak{F}$).

The most promising approaches to circumvent such difficulties came from the identification of the algebraic structure underlying the problem. Reference [27] resorts first to the polar decomposition of step operators in $u(2) \equiv u(1) \otimes su(2)$ and then to its contraction

[1] It is known that the commutation relation $[\hat{\phi}, \hat{n}] = i$ can be implemented as an operator identity only on a dense set of the Hilbert space. See [14].

limit. Being $su(2)$ compact, the closure property in this way is lost. At this point the idea of using $su(1,1)$, a non-compact counterpart of $su(2)$, was introduced [21, 28], based on relative number variables, well-defined (e.g. in the frame of thermo-field dynamics) in a two-mode representation of $su(1,1)$. The crucial notion furnished by these approaches was just the realization of the role of such algebra. This notion was recently revived. Observing that in a canonical quantization scheme the self-adjoint Lie algebra generators K_1, K_2, K_3 of the group $SO^{\uparrow}(1,2)$ correspond to the classical polar coordinates variables in $\mathbb{R}^2$, $\mathcal{R}_x \doteq r\cos\phi$, $\mathcal{R}_y \doteq r\sin\phi$ and $\mathcal{R}_z \doteq r$, respectively, whose Poisson brackets ($\{\mathcal{R}_x, \mathcal{R}_y\}_{PB} = \mathcal{R}_z$, $\{\mathcal{R}_z, \mathcal{R}_x\}_{PB} = -\mathcal{R}_y$, $\{\mathcal{R}_z, \mathcal{R}_y\}_{PB} = \mathcal{R}_x$) satisfy the same algebra, Kastrup [29] proposed a group theoretical approach to the problem resorting to the irreducible unitary representations of the positive series.

The generators K_1, K_2, K_3 have commutation relations $[K_1, K_2] = -\mathrm{i}K_3$, $[K_2, K_3] = \mathrm{i}K_1$, $[K_3, K_1] = \mathrm{i}K_2$, or, introducing the skew raising and lowering operators $K_\pm \doteq K_1 \pm \mathrm{i}K_2$ acting as ladder operators on the eigenvectors of the Cartan operator K_3, generator of the compact subgroup $SO(2)$ of $SO^{\uparrow}(1,2)$, $[K_+, K_-] = -2K_3$, $[K_3, K_\pm] = \pm K_\pm$.

The positive discrete series representation of the algebra $su(1,1)$ of the double covering of $SO^{\uparrow}(1,2)$ is characterized [30] by the existence of an highest weight vector $|\kappa, \Omega\rangle$ (invariant, in representation space, under the action of the maximal compact sub-group $\mathcal{K} \sim U(1)$ of $SU(1,1)$),[2] annihilated by the lowering operator K_-. This means that $K_-|\kappa, \Omega\rangle$ is *not* a ket in the representation Hilbert space. Upon identifying $|\kappa, \Omega\rangle$ with the eigenvector $|\kappa, 0\rangle$ of K_3 the commutation relations give $K_-|\kappa, 0\rangle \equiv 0$ (see below). The real number κ, characterizing the representation, assumes values $\kappa = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and is such that the Casimir operator $\mathcal{C}_2 = \frac{1}{2}\{K_+, K_-\} - K_3^2$ has in that representation eigenvalue $\kappa(1-\kappa)$. The positive discrete series irreducible unitary representation $\mathcal{D}_\kappa^{(+)}$, with $K_-^\dagger \equiv K_+$, is spanned by the complete orthonormal set $\{|\kappa, n\rangle | n \in \mathbb{N}\}$ of eigenstates of K_3, $K_3|\kappa, n\rangle = (n+\kappa)|\kappa, n\rangle$. The ladder operators K_-, K_+ satisfy the equations

$$
\begin{aligned}
K_-|\kappa, n\rangle &= \omega_n^{-1}\sqrt{n(n+2\kappa-1)}|\kappa, n-1\rangle, \\
K_+|\kappa, n\rangle &= \omega_{n+1}\sqrt{(n+1)(n+2\kappa)}|\kappa, n+1\rangle,
\end{aligned}
\tag{4}
$$

where $\omega_n = \mathrm{e}^{\mathrm{i}\theta_n}$ is a phase. With no loss of generality θ_n can be assumed independent on n ($\omega_n = \mathrm{e}^{\mathrm{i}\theta} \doteq \omega$, $\forall n \in \mathbb{N}$). Notice that equations (4) imply

$$
|\kappa, n\rangle = \omega^{-n}\frac{1}{n!}\binom{n+2\kappa-1}{2\kappa-1}^{-\frac{1}{2}} K_+^n|\kappa, 0\rangle.
$$

$\mathcal{D}_\kappa^{(+)}$ has an interesting realization in $h(1)$, the so called Holstein–Primakoff [31] representation: in it the basis vector $|\kappa, n\rangle$, eigenvector of K_3, is identified with the eigenvector $|n\rangle$ of $\hat{n}$ in $\mathfrak{F}$, namely $K_3 \doteq \hat{n} + \kappa$, then, over $\mathfrak{F}$,

$$
K_+ \equiv \omega\sqrt{\hat{n}+2\kappa-1}a^\dagger, \qquad K_- \equiv \omega^{-1}a\sqrt{\hat{n}+2\kappa-1}.
$$

For $\kappa = \frac{1}{2}$, in this realization $\mathcal{D}_{\frac{1}{2}}^{(+)}$ is generated by

$$
K_+ \equiv \omega\sqrt{\hat{n}}a^\dagger, \qquad K_- \equiv \omega^{-1}a\sqrt{\hat{n}}, \qquad K_3 \equiv \hat{n} + \tfrac{1}{2}.
$$

namely $\left|\frac{1}{2}, n\right\rangle = \omega^{-n}\frac{1}{n!}K_+^n\left|\frac{1}{2}, 0\right\rangle$.

There is a connection between the latter representation of $su(1,1)$ and the uniform expansion which (3) reduces to for $\ell = 0$, which is at the basis of the lack of convergence mentioned above: indeed a uniform superposition of occupation number eigenstates in $\mathfrak{F}$ can

[2] It is worth recalling that besides the discrete series, one of the principal continuous series (class I) and the supplementary series have this property.

be straightforwardly constructed in $\mathcal{D}^{(+)}_{\frac{1}{2}}$, in view of the form of $\left|\frac{1}{2}, n\right\rangle$, as $|\omega\rangle \doteq \mathrm{e}^{\omega K_+}\left|\frac{1}{2}, 0\right\rangle = \sum_{n=0}^{\infty}\left|\frac{1}{2}, n\right\rangle$, a coherent state of $su(1,1)$ which is manifestly *not* normalizable.

For analogy with the classical case ($\cos\phi = r^{-1}\mathcal{R}_x$, $\sin\phi = r^{-1}\mathcal{R}_y$) in [29] the definition is suggested:

$$\widehat{\cos\phi} \equiv \tfrac{1}{2}\left\{K_3^{-1}, K_1\right\}, \qquad \widehat{\sin\phi} \equiv \tfrac{1}{2}\left\{K_3^{-1}, K_2\right\}. \tag{5}$$

In the discrete positive series irreducible unitary representation of $su(1,1)$, this assumption is analytic for generic κ, because K_3^{-1} is well-defined in view of the eigenvalue equation of K_3, but it has the drawback that $\widehat{\cos\phi}^2 + \widehat{\sin\phi}^2$ and $[\widehat{\cos\phi}, \widehat{\sin\phi}]$, even though both diagonal, are not equal—respectively—to $\mathbb{I}$ and 0, as they should be. Indeed, they do satisfy such conditions only for large n:

$$\left(\widehat{\cos\phi}^2 + \widehat{\sin\phi}^2\right)|\kappa, n\rangle = \frac{1}{8}\left[\left(f_{n+1}^{(\kappa)}\right)^2 + \left(f_n^{(\kappa)}\right)^2\right]|\kappa, n\rangle,$$

$$[\widehat{\cos\phi}, \widehat{\sin\phi}]|\kappa, n\rangle = \frac{1}{8\mathrm{i}}\left[\left(f_{n+1}^{(\kappa)}\right)^2 + \left(f_n^{(\kappa)}\right)^2\right]|\kappa, n\rangle,$$

where

$$f_n^{(\kappa)} \doteq [n(n+2\kappa-1)]^{\frac{1}{2}}\left(\frac{1}{n+\kappa} + \frac{1}{n+\kappa-1}\right) \underset{n\to\infty}{\longrightarrow} 2.$$

In this note we suggest that the scheme (5) can be generalized so as to exhibit almost everywhere in $\mathfrak{F}$ (i.e., with the exception of point $|0\rangle$) all the required features for effective phase-angle quantum variables.

The basic step is the generalization of equations (5) (see also [32, 33]):

$$\widehat{\cos\phi} \doteq \{\mathcal{F}(K_3), K_1\}, \qquad \widehat{\sin\phi} \doteq \{\mathcal{F}(K_3), K_2\}, \tag{6}$$

where $\mathcal{F}$ is a function—whose existence will be proved in the following—meromorphic in the $\mathcal{D}^{(+)}_{\frac{1}{2}}$-representation, of the form $\mathcal{F}(K_3) = \hat{n}^{-1}(\mathbb{I} - |0\rangle\langle 0|) + \Phi(\hat{n})$, where $\Phi(\hat{n})$, whose eigenvalues in $\mathfrak{F}$ will be written in terms of digamma functions, is such that $\Phi(\hat{n})|0\rangle = 0$. Such choice guarantees that the commutation relations $[\hat{n}, \widehat{\cos\phi}] = -\mathrm{i}\,\widehat{\sin\phi}$, $[\hat{n}, \widehat{\sin\phi}] = \mathrm{i}\,\widehat{\cos\phi}$, are satisfied $\left(\hat{n} = K_3 - \frac{1}{2}\right)$ and $\mathcal{F}$ may be selected so as to ensure that the two other conditions are satisfied almost everywhere (i.e. in $\mathfrak{F}\backslash|0\rangle$). The construction of $\mathcal{F}$ in $\mathcal{D}^{(+)}_{\frac{1}{2}}$ in such a way that:

$$\text{(i)}\quad \widehat{\cos\phi}^2 + \widehat{\sin\phi}^2 = \mathbb{I}, \qquad \text{(ii)}\quad [\widehat{\cos\phi}, \widehat{\sin\phi}] = 0, \tag{7}$$

proceeds as follows. One observes first that, upon setting $\mathfrak{c} \equiv \widehat{\cos\phi}$, $\mathfrak{s} \equiv \widehat{\sin\phi}$ and $\mathcal{L}_n \doteq \frac{1}{2}n\left(\mathcal{F}\left(n - \frac{1}{2}\right) + \mathcal{F}\left(n + \frac{1}{2}\right)\right)$,

$$\mathfrak{c}|n\rangle = \omega\mathcal{L}_{n+1}|n+1\rangle + \omega^{-1}\mathcal{L}_n|n-1\rangle, \qquad \mathfrak{s}|n\rangle = -\mathrm{i}\omega\mathcal{L}_{n+1}|n+1\rangle + \mathrm{i}\omega^{-1}\mathcal{L}_n|n-1\rangle,$$

where $|n\rangle \equiv \left|\frac{1}{2}, n\right\rangle$. Equivalently,

$$\mathfrak{e}_{\pm} \doteq (\mathfrak{c} \pm \mathrm{i}\mathfrak{s}) \equiv \widehat{\mathrm{e}^{\pm\mathrm{i}\phi}}, \qquad \mathfrak{e}_{\pm} = \{\mathcal{F}(K_3), K_{\pm}\},$$

satisfy

$$\mathfrak{e}_+|n\rangle = 2\omega\mathcal{L}_{n+1}|n+1\rangle, \qquad \mathfrak{e}_-|n\rangle = 2\omega^{-1}\mathcal{L}_n|n-1\rangle.$$

From these, for

$$\mathcal{O} \doteq \tfrac{1}{4}\xi[(1+\eta)\mathfrak{c} + \mathrm{i}(1-\eta)\mathfrak{s}][(1+\xi\eta)\mathfrak{c} - \mathrm{i}(1-\xi\eta)\mathfrak{s}],$$

where $\xi, \eta \in \{\pm 1\}$, $(\mathcal{O}_{1,1} \equiv \mathfrak{c}^2, \mathcal{O}_{1,-1} \equiv \mathfrak{s}^2, \mathcal{O}_{-1,1} \equiv \mathrm{i}\mathfrak{c}\mathfrak{s}, \mathcal{O}_{-1,-1} \equiv -\mathrm{i}\mathfrak{s}\mathfrak{c})$, one finds

$$\mathcal{O}_{\xi,\eta}|n\rangle = \left[\mathcal{L}_{n+1}^2 + \xi\mathcal{L}_n^2\right]|n\rangle + \eta(\omega^2\mathcal{L}_{n+1}\mathcal{L}_{n+2}|n+2\rangle + \xi\omega^{-2}\mathcal{L}_{n-1}\mathcal{L}_n|n-2\rangle),$$

whence $(\mathfrak{c}^2 + \mathfrak{s}^2)|n\rangle = 2\left[\mathcal{L}_n^2 + \mathcal{L}_{n+1}^2\right]|n\rangle$, and $[\mathfrak{c}, \mathfrak{s}]|n\rangle = 2\mathrm{i}\left[\mathcal{L}_{n+1}^2 - \mathcal{L}_n^2\right]|n\rangle$. The pursued conditions (7) amount then to requiring that $\mathcal{L}_{n+1} = \mathcal{L}_n = \frac{1}{2}$, for all $n \geqslant 1$, namely that the following system of recursion equations is satisfied by $F(n) \doteq \mathcal{F}\left(n + \frac{1}{2}\right)$:

$$(n+1)F(n+1) - nF(n-1) + F(n) = 0, \tag{8}$$

$$F(n) = \frac{1}{n} - F(n-1). \tag{9}$$

Equations (9) and (8) are mutually consistent; thus we only need to consider, e.g., (8). Equation (9) will simply be used to properly continue the result to $n = 0$. For $n \geqslant 1$, the recursion equation (8) can be dealt with by the generating function method. One defines first

$$f(z) \doteq \sum_{n=1}^{\infty} F(n)z^n, \tag{10}$$

where z is an undeterminate. Multiplying equation (8) by z^n and summing for n ranging from 1 to ∞, in view of definition (10), one obtains for f the differential equation

$$(1 - z^2)\frac{\mathrm{d}f}{\mathrm{d}z} + (1 - z)f(z) - F(1) = 0. \tag{11}$$

We select $F(1) = 1$, so that, as only one integration constant is required in view of (9), we may assume $F(0) \equiv f(0) = 0$. The solution of (11) is then

$$f(z) = -(1 + z)^{-1}\ln(1 - z). \tag{12}$$

Expanding (12) as a power series in z one eventually finds

$$F(n) = (-)^n \sum_{\ell=1}^{n} \frac{(-)^\ell}{\ell}, \qquad \forall n \geqslant 1. \tag{13}$$

The sum in (13) can be split into even and odd values of ℓ, and has different expressions depending on whether n is itself even or odd:

$$F(2m) = \frac{1}{2}\sum_{r=1}^{m}\frac{1}{r} - \sum_{r=0}^{m-1}\frac{1}{2r+1}, \qquad m \geqslant 1,$$

$$F(2m+1) = -\frac{1}{2}\sum_{r=1}^{m}\frac{1}{r} + \sum_{r=0}^{m}\frac{1}{2r+1}, \qquad m \geqslant 0.$$

Recalling now that [34] $\sum_{\ell=1}^{k}\frac{1}{\ell} = \gamma + \psi(k+1)$, and $\sum_{\ell=0}^{k-1}\frac{1}{2\ell+1} = \frac{1}{2}\left(\gamma + 2\ln 2 + \psi\left(k + \frac{1}{2}\right)\right)$, where γ is Euler's constant and $\psi(z)$ the Digamma function, and that $\frac{1}{2}\psi\left(k + \frac{1}{2}\right) = \psi(2k) - \frac{1}{2}\psi(k) - \ln 2$, one obtains—including as well the condition $F(0) = 0$, and denoting by $[\![x]\!]$ the largest integer $\leqslant x$—

$$\begin{aligned} F(n) &= (1 - \delta_{n,0})\frac{1}{n} + \Phi(n), \\ \Phi(n) &= (-)^n\left[\psi\left(\left[\!\!\left[\frac{1}{2}(n+1)\right]\!\!\right]\right) - \psi(n)\right]. \end{aligned} \tag{14}$$

We may then conclude that the function entering definition (6) $\mathcal{F}(K_3) \equiv F(\hat{n})$, does exist and is well defined as operator[3] over the Fock space $\mathfrak{F}$ for $n \geqslant 1$. With this choice, indeed,

$$\mathfrak{e}_+|n\rangle = \omega|n+1\rangle, \qquad n \geqslant 0; \mathfrak{e}_-|n\rangle = \omega^{-1}|n-1\rangle, \qquad n \geqslant 1.$$

Still, what happens in state $|0\rangle$ has some ambiguity and subtlety that requires further discussion. Explicit calculation using the above scheme shows that while the conditions (7) hold for all $|n\rangle$ with $n \geqslant 1$, and in state $|n\rangle$, for which operator $\hat{n}$ is sharp, give rise to the 'semi-classical' uncertainty condition $\Delta_n(\mathfrak{c})\Delta_n(\mathfrak{s}) = \frac{1}{2}$, namely the minimum variance for $\mathfrak{s}$ and $\mathfrak{c}$ consistent with $\mathfrak{c}^2 + \mathfrak{s}^2 = \mathbb{I}$, for state $|0\rangle$ one has $(\mathfrak{c}^2 + \mathfrak{s}^2)|0\rangle = \frac{1}{2}|0\rangle, [\mathfrak{c}, \mathfrak{s}]|0\rangle = -\frac{1}{2}\mathrm{i}|0\rangle$ and $\mathfrak{e}_-|0\rangle = 0$, as well as the 'sub-minimal' uncertainties $\Delta_0(\mathfrak{c}) = \Delta_0(\mathfrak{s}) = \frac{1}{2}$, suggesting that $\mathfrak{c}$ and $\mathfrak{s}$ are still not uncorrelated quantum observables there. This is the 'fossil remain' in our representation of the pathology encountered in all other representations and the price paid for making independent on n the eigenvalues of $(\mathfrak{c}^2 + \mathfrak{s}^2)$ and $[\mathfrak{c}, \mathfrak{s}]$. Indeed, the action of $\mathfrak{e}_-$ is badly defined in state $|0\rangle$: one finds $\mathfrak{e}_+\mathfrak{e}_- = \mathbb{I} - |0\rangle\langle 0|, \mathfrak{e}_-\mathfrak{e}_+ = \mathbb{I}$, namely $[\mathfrak{e}_-, \mathfrak{e}_+] = |0\rangle\langle 0|$, the same disease as in (2), even though within a much more robust structure, where sine and cosine operators satisfy the desired conditions $\mathfrak{c}^2 + \mathfrak{s}^2 = \mathbb{I}, [\mathfrak{c}, \mathfrak{s}] = 0$ in all of $\mathfrak{F}\backslash|0\rangle$. In view of the commutation relation $[\hat{\phi}, \hat{n}] = \mathrm{i}$, of course in a state which is sharp for $\hat{n}$, the value of $\hat{\phi}$ should be fully undeterminate; yet the uncertainty product $\Delta_0(\mathfrak{c})\Delta_0(\mathfrak{s}) = \frac{1}{4}$ shows that its sine and cosine have mutual uncertainty when $\hat{n}$ has eigenvalue 0: this means that even in the quantum case the phase of the origin is undefined [25] (what is the longitude of the north pole?). A suggestive hint comes from the property that in our reference algebra the defining relations no longer contract in state $|0\rangle$ to those of $E(2)[K_\alpha \mapsto \bar{K}_\alpha, \alpha = +, -, 3, [\bar{K}_3, \bar{K}_\pm] = \pm\bar{K}_\pm, [\bar{K}_+, \bar{K}_-] = 0]$, as they do instead in $|n\rangle; n \geqslant 1$.

There is however yet another new feature in the present approach. In the form (14), $F(n)$ can be analytically continued to $n = -1$, where, in view of the identities [34] $\psi(z+1) = \psi(z) + \frac{1}{z}$, $\psi(1-z) = \psi(z) + \pi\cot(\pi z)$, one finds $F(-1) = 0$. Notice that this is not consistent with (9), which however obviously does not hold for $n = 0$. For $n = 0$ a more plausible solution to (9) would instead demand

$$\lim_{n\to 0} nF(n-1) = 1. \tag{15}$$

This leads to an intriguing possible way out of the difficulty, which consists of replacing $\Phi(n)$ in (14) by

$$\Psi(n) = \Phi(n) + (-)^n \sum_{k=1}^{\infty} \delta_{n,-k}\zeta(k), \tag{16}$$

where $\zeta(1) = -\omega^{-1}\psi(-1)$ and, $\forall k \geqslant 2$,

$$\zeta(k) \doteq \zeta(1) + \psi\left(\left[\!\left[\frac{k}{2}\right]\!\right]\right) - \psi\left(2\left[\!\left[\frac{k}{2}\right]\!\right]\right) + \frac{1-(-)^k}{2(k-1)}.$$

Indeed, with this new self-adjoint operator $\Psi(\hat{n}), F(\hat{n})$, defined now as $F(\hat{n}) = \hat{n}^{-1}\sum_{m=1}^{\infty}|m\rangle\langle m| + \Psi(\hat{n})$,[4] leads to $\mathfrak{e}_+\mathfrak{e}_- = \mathbb{I}, \mathfrak{e}_+ = \mathfrak{e}_-^\dagger$ and $[\mathfrak{e}_-, \mathfrak{e}_+] = 0$ also in state $|0\rangle$. The rationale here is the extension of the space of states $\mathfrak{F}$ to include the extra vector '$|-1\rangle$' ($|-1\rangle \doteq \mathfrak{e}_-|0\rangle \equiv \mathcal{F}(K_3)K_-|0\rangle$): assuming $K_-|-\ell\rangle = \alpha_{-\ell}|-\ell-1\rangle$, with $\alpha_{-\ell}$ finite $\forall\ell \geqslant 1$

[3] Over $\mathfrak{F}$, $[\![x]\!]$ can be realized in terms of multiboson operators $A_k, A_k^\dagger, \hat{N}_k \doteq A_k^\dagger A_k$ [35], which close an algebra $h(1)$, with $[A_k, \hat{n}] = kA_k$. For $n \equiv uk + v$, where $u \doteq [\![\frac{n}{k}]\!]$, and $v \doteq \{n\}_k$ is the residue of $n(\mathrm{mod}\,k)$, one has $\hat{N}_k|n\rangle = u|n\rangle$. $\psi(\hat{n})$ can be constructed resorting to the integral representation $\psi(\hat{n}) = \int_0^\infty \mathrm{d}x\,\frac{\mathrm{e}^{-x} - \mathrm{e}^{x\hat{n}}}{1-\mathrm{e}^{-x}} - \gamma$ [34].

[4] In Fock space operator $\sum_{m=1}^{\infty}|m\rangle\langle m|$ is diagonal in the basis state $|n\rangle$ ($\forall n \geqslant 0$) in which it has eigenvalue $(1-\delta_{n,0})$.

(notice that $\alpha_0 = 0$: the sufficient requirement is simply that K_- applied to $|0\rangle$ generates a vector lying *outside* $\mathfrak{F}$, but the coefficient is retained as zero, which however is to be multiplied by the infinite value of the ψ function in F at its pole in -1 when the action of $\mathfrak{e}_-$ is considered), then $\mathfrak{e}_-|-\ell\rangle = 0|\ell-1\rangle$, $\forall \ell \geqslant 1$. (This implies that in fact in equation (16) only the firt two terms in the sum ($k = 1, 2$) could be retained with no loss of generality.) It is interesting to observe that $[\mathfrak{e}_-, \mathfrak{e}_+] = 0$ also in any state $|-\ell\rangle$ with $\ell \geqslant 2$, whereas, in $|-1\rangle$, $[\mathfrak{e}_-, \mathfrak{e}_+] = -1$, namely in the new state introduced the abelian algebra generated in $\mathfrak{F}$ by $\mathfrak{e}_-$, $\mathfrak{e}_+$ expands to $h(1)$.

Ket $|-1\rangle$ has the correct normalization property $\langle -1|-1\rangle = 1$, and is obviously orthogonal to any vector in $\mathfrak{F}$: $\langle n|-1\rangle = 0, \forall n \geqslant 0$, leading to the existence [36] of both a PVM and a POM in $\mathcal{F}$.

The construction of space $\mathcal{H} \doteq \mathfrak{F} \cup |-1\rangle$ can be consistently and rigorously carried over resorting to the notion of 'dilated extension Hilbert space' [38–40], typically encountered when one has to construct Hilbert spaces from certain given data. An efficient pattern of such constructions is the lemma known as Kolmogorov's dilation theorem for positive definite kernels, and Sz-Nagy's dilation theorem. The spectral theory for self-adjoint operators shows indeed that one may generally restrict the attention to bounded operators by the use of Cayley transform: if the operator is only symmetric, possibly with dense domain, but not self-adjoint, then the Cayley transform reduces the problem to the analysis of partial isometries. But if the index is nonzero (i.e. the co-dimension of the initial and the final space are unequal) then there will not be self-adjoint extensions in the same, given, Hilbert space, but an induced extension Hilbert space is needed for a complete understanding of spectral resolutions and self-adjoint operator extensions of unbounded operators. This, due to the Gel'fand–Naimark theorem [41], holds for Lie algebras only in the case they are simple, and requires passing to the universal enveloping algebra to get the appropriate $*$-representation [42].

From the operational point of view, as in the present case co-dimension 1 is sufficient, a plausible way to perform the extension is to recall that the supplementary series representation of $SU(1,1)$, already mentioned, is spanned by a basis $\{|\lambda,\mu\rangle\}$, corresponding to $\mathcal{C}_2 = |\lambda|^2 + \frac{1}{4}$, where λ is pure imaginary: $\lambda = \mathrm{i}\tau$, $-\frac{1}{2} < \tau < \frac{1}{2}$, such that

$$K_3|\lambda,\mu\rangle = \mu|\lambda,\mu\rangle, \mu = 0, \pm 1, \pm 2, \ldots,$$
$$K_\pm|\lambda,\mu\rangle = \left(\pm\left(\tfrac{1}{2} - \mathrm{i}\lambda\right) + \mu\right)|\lambda,\mu\pm 1\rangle.$$

These expressions imply that for $\lambda(\tau) \to 0$, which leads just to $\mathcal{C}_2 = \frac{1}{4}$, such representation is in correspondence (one-to-one, for $\mu \geqslant 0$, up to a shift of the Cartan operator eigenvalue $\mu \leftrightarrow n + \frac{1}{2}$) with the positive discrete series representation. Construction (16) indeed guarantees that the two irreps can be made to correspond in a one-to-one way. Ket $|-1\rangle$, necessary for the Hilbert space dilation extension, may be thus thought of as 'borrowed' from the supplementary series[5].

We conclude conjecturing that the technical way in which the corresponding intertwining operation can be realized, is but an analytic continuation from one representation to the other that can be implemented by resorting to the q-deformed algebra $su_q(1,1)$.[6] The

[5] In the supplementary series, for $\tau = \frac{1}{2}-$, $K_\pm|\lambda,\mu\rangle = (\mu \pm 1)|\lambda,\mu\pm 1\rangle$, $K_3|\lambda,\mu\rangle = \mu|\lambda,\mu\rangle$, identifying $\mu < 0$ as the number of quanta removed from the vacuum. On the other hand, explicit calculation shows that $K_3|-1\rangle = -\frac{1}{2}|-1\rangle$. The 'extension' state $|-1\rangle$ would thus be characterized in the deformed Fock space by $n = -1$, if interpreted in the positive discrete series representation, as expected. Interpreting instead the new state as belonging to supplementary series, such a state would correspond to 'energy' $(\mu + \frac{1}{2})$ equal to $-\frac{1}{2}$, which corresponds to removing a quantum from the physical vacuum (which has 'zero point energy' equal to $\frac{1}{2}$).

[6] For a discussion of the role of deformed algebras in accounting for the Josephson junction energy spectrum see [39].

latter is defined—in the universal enveloping algebra of $SU(1,1)$—in terms of generators $\{Q_-, Q_+, Q_3\}$ by [43]

$$[Q_3, Q_\pm] = \pm Q_\pm, [Q_+, Q_-] = -[\![2Q_3]\!]_q,$$

where the q-deformation ($q \in \mathbb{C}$) is as customary given by $[\![x]\!]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}$. The q-deformed Casimir invariant is $\mathcal{C}_2^{(q)} = Q_+Q_- - [\![Q_3]\!]_q[\![Q_3 - 1]\!]_q$ (attention should be paid to the different meaning of the symbol $[\![\bullet]\!]$, which however is distinguished here by the subscript q).

In terms of the generators of $SU(1,1)$, $su_q(1,1)$ has presentation [44–46],

$$Q_3 = K_3, \qquad Q_- = K_- g(K_3), \qquad Q_+ = g(K_3)K_+,$$

with[7]

$$g(K_3) = \sqrt{\frac{[\![K_3 - \kappa]\!]_q[\![K_3 + \kappa - 1]\!]_q}{(K_3 - \kappa)(K_3 + \kappa - 1)}},$$

for $\mathcal{C}_2^{(q)} = [\![\kappa]\!]_q[\![1 - \kappa]\!]_q$.

Note that for $q = \mathrm{e}^{4\lambda} \equiv \mathrm{e}^{4\mathrm{i}\tau}$ and $\kappa = \frac{1}{2}$, $\mathcal{C}_2^{(q)} = \frac{1}{4}(1 - 2\sin^2\tau)^{-2}$. Then for $|\lambda| \ll \frac{1}{2}$, up to $\mathcal{O}(|\lambda|^4)$, $\mathcal{C}_2^{(q)} \approx \frac{1}{4} + |\tau|^2 \equiv \frac{1}{4} + |\lambda|^2$, as desired. The latter gives rise to the representation $Q_3|\kappa, n\rangle = (n+\kappa)|\kappa, n\rangle$ (diagonal), $Q_+|\kappa, n\rangle = \omega\sqrt{[\![n+1]\!]_q[\![n+2\kappa]\!]_q}|\kappa, n+1\rangle$, $Q_-|\kappa, n\rangle = \omega^{-1}\sqrt{[\![n]\!]_q[\![n+2\kappa-1]\!]_q}|\kappa, n-1\rangle$.

This construction also accounts for the contraction to $h(1)$ in state $|-1\rangle$.

Physically we suggest the interpretation that whereas $|0\rangle$, even though representing a physical ‘vacuum’, namely a state with no excitations (zero occupation number), yet is still in the space of states of the system, $|-1\rangle$ does instead describe the true vacuum, i.e. ‘emptiness’ (no states of the system exist), out of whose quantum fluctuations Fock space states (in particular $|0\rangle$) can be generated (notice, by $\mathfrak{e}_+$, *not* by K_+). Of course, in any realistic model of a physical system whose Hamiltonian operator contains—such as, e.g., the Hamiltonian describing systems with Josephson coupling—terms in $\widehat{\cos\phi}$ or $\widehat{\sin\phi}$, the latter should in general be replaced by their orthogonal projection from $\mathcal{H}$ onto $\mathfrak{F}$.

Applications of the scheme proposed are numerous, and some may indeed lead to unexpected results. The motion of Wannier particles—electrons and holes or Cooper pairs—constrained to ‘hop’ from a site to neighbouring sites on a lattice [47, 48], such as in the Hubbard (and Bose–Hubbard) and related models; the still controversial phenomenon of Bloch oscillations for strongly correlated exciton motion in crystals, where the oscillations are found to be damped, in sharp contrast with the predictions of the semi-classical solid state model, besides the already mentioned problem of the spectral and transport properties of ‘fully quantum’ Josephson junctions, constitute just a sample. The approach here proposed may lead to a more correct understanding of a number of questions: for example, the Bloch oscillations damping might plausibly be due to the existence of the ‘sink’ state $|-1\rangle$.

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[7] Manifestly, $Q_+ = Q_-^\dagger$. Moreover, for $q \to 1$, $Q_\alpha \to K_\alpha$, $\alpha = +, -, 3$ $(g(K_3) \to 1)$ and $\mathcal{C}_2^{(q)} \to \mathcal{C}_2$.

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